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SYMMETRIC PRESENTATION OF LINK MODULES

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The Alexander Module of a one-dimensional link, when suitably localized, is determined by a matrix which expresses the longitudes as linear combinations of the meridians. It is shown here that this (longitude) matrix may be chosen to satisfy a Hermitian property. In this Hermitian context, we determine the change in the longitude matrix produced by a surgery on the link.

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classical link, link module, Alexander module, localization, Alexander polynomial, longitude matrix

In [7] and [4] it is pointed out that the Alexander module A of a classical link K becomes considerably simpler if its ring $\Lambda = Z[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}]$ (m = number of components in the link) is *completed* [7] to $\hat{\Lambda}$, using the I-adic filtration, where I is the *augmentation* ideal $(t_1 - 1, \ldots, t_m - 1)$, or localized [7] to Λ_S with respect to the multiplication set $S = \{\lambda(t_1, \ldots, t_m): \lambda(1, \ldots, 1) = \pm 1\}$. In fact $\hat{A}$ and A_S have the geometric property of being invariants of the topological I-equivalence class of K. Although A_S obviously determines $\hat{A}$, they are equivalent weakenings of A—i.e. given A, A', then $\hat{A} \approx \hat{A}'$ implies $A_S \approx A'_S$. The advantage in considering A_S is in the realization problem. Given a module B over $\hat{\Lambda}$, it is difficult to decide whether $B \approx \hat{A}$ for some finitely-generated Λ-module A. The analogous question over Λ_S is trivial.

In [7] it is also pointed out that $\hat{A}$ and A_S have presentations of the following form: there are generators $\mu_1, \ldots, \mu_m$, represented by meridians of the different components, and elements $\lambda_1, \ldots, \lambda_m$, represented by longitudes, such that:

(1) $\lambda_i = \sum_{j=1}^m a_{ij}\mu_j$ $(i = 1, \ldots, m)$, $a_{ij} \in \Lambda_S$ (or $\hat{\Lambda}$).

(2) A presentation of A_S (or $\hat{A}$) is given by

$$\left\{\mu_1, \ldots, \mu_m: (T_i - 1)\mu_i = (t_i - 1)\sum_j a_{ij}\mu_j\right\}.$$

$$T_i = t_1^{l_{i1}} \cdots t_m^{l_{im}},$$

where $l_{ij} = a_{ij}(1, \ldots, 1)$ is the linking number of the ith and jth components (l_{ii} can be specified arbitrarily, but is usually taken to be zero).

The matrix $a=(a_{ij})$ is not uniquely determined. It will depend upon:

(i) The choice of $\{\mu_i, \lambda_i\}$,

(ii) The ambiguity due to the relations of (2).

This indeterminacy is analogous to the indeterminacy of the Milnor $\bar{\mu}$-invariants and, in fact, a may be regarded as a "symmetrization" of the $\bar{\mu}$-invariants. (See [6, Theorem A].)

This paper has two primary purposes:

(I) We demonstrate that a may be chosen to satisfy a certain Hermitian property. To simplify the exposition we restrict ourselves to the case $l_{ij}=0$, for all i, j, although there is a, more complicated, general result.

It seems reasonable to conjecture that this Hermitian property (in addition to the condition: $\sum_j a_{ij}(t_j-1)=0$—see below) is the only constraint on a. This raises the problem of realizing any such a by a link.

We will obtain, as a by-product of (I), that the Alexander module A_S, for a 2-component link, is determined (in an explicit way) by the Alexander polynomial.

(II) As a step toward a general realization result we determine the effect on a of a surgical modification of the link (as in [5], [1], [9]).

In [6] we give an algebraic criterion for two links to be equivalent under the equivalence relation generated by surgical modification. In the special case $l_{ij}=0$, this will amount to requiring that the Milnor $\bar{\mu}$-invariants $\bar{\mu}_{ijk}$ be equal for every i, j, k. Thus the realization problem for A_S is reduced to determining the Alexander modules for a representative link for each choice of $\{\mu_{ijk}\}$. For example, in the 3-component case, we can take the Borromean rings, replacing one of the components with a cable running around it n times, for any $n \in Z$ (see [8]).

One particular consequence of our point of view in studying A_S instead of A, is that the problem of characterizing the Alexander polynomial of a link seems approachable again. In [2] a counter-example is given to the obvious conjecture, based upon the Torres conditions, of which polynomials can be the Alexander polynomial of a link. The nature of this counter-example makes it difficult to make a new conjecture. However it is easy to see that the argument of [2] does not give a counter-example in the context of the *localized* polynomial. In fact we show that, for 2-components, any polynomial satisfying the Torres conditions is the localized Alexander polynomial of a link. This uses only the results of [1].

We recommend [3] as a good general reference.

§ 1

Let K be a link (smooth imbedding of ordered, oriented, disjoint circles in the 3-sphere) of n components: $K_1, \ldots, K_n$ the (oriented) components of K. Let N_i be a tubular neighborhood of K_i, with $N_i \cap N_j=\emptyset$ for every $i \neq j$. Choose a basepoint $x_i \in \partial N_i$, and meridian and longitude circles $m_i, l_i \subset \partial N_i$, intersecting at x_i, oriented so that in ∂N_i the intersection numbers $m_i \cdot l_i=+1$. If it is required that l_i be oriented in the same direction as K_i and the linking number of K_i and l_i be 0, the isotopy classes of m_i, l_i in ∂N_i are well-defined.

Let x_0 be a basepoint in the interior of $X = \overline{S^3 - \bigcup_i N_i}$. Let $A = A(K) = H_1(\tilde{X}, \tilde{x}_0)$ where $\tilde{X} \to X$ is the universal abelian covering and $\tilde{x}_0$ is the fiber over x_0. Note that $H_1(X) \approx Z^n$ is the group of covering translations of $\tilde{X} \to X$. A basis $t_1, \dots, t_n$ for $H_1(X)$ is defined by $m_1, \dots, m_n$—we identify the group ring $Z[H_1(X)]$ with $\Lambda = Z[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. $A = A(K)$ is then a module over Λ referred to as the *Alexander module* of K.

An important additional structure on A is the homomorphism $e: A \to \Lambda$ defined by the boundary homomorphism $H_1(\tilde{X}, \tilde{x}_0) \to H_0(\tilde{x}_0)$. $H_0(\tilde{x}_0)$ is identified with Λ by choice of a basepoint $\bar{x}_0 \in \tilde{x}_0$. See [6, p. 49].

We single out certain important elements of A. Choose oriented arcs a_i from x_0 to x_i in X. Assuming a chosen basepoint $\bar{x}_0 \in \tilde{x}_0$—then a_i, m_i, l_i lift to arcs $\bar{a}_i$, $\bar{m}_i$, $\bar{l}_i$, where $\bar{a}_i$ starts at $\bar{x}_0$ and $\bar{m}_i$, $\bar{l}_i$ start at the end point $\bar{x}_i$ of $\bar{a}_i$. Note that $\bar{m}_i$ ends at $t_i(\bar{x}_i)$ and $\bar{l}_i$ ends at $T_i(\bar{x}_i)$—recall T_i defined in the introduction ($l_{ii} = 0$). Define elements $\mu_i, \lambda_i \in A$ to be the homology classes of the chains: $\bar{m}_i + (1 - t_i)\bar{a}_i$, $\bar{l}_i + (1 - T_i)\bar{a}_i$. Note that $e(\mu_i) = t_i - 1$, $e(\lambda_i) = T_i - 1$.

A link K equipped with arcs $\{a_1, \dots, a_n\}$, a_i running from a basepoint x_0 to K_i, will be referred to as a *based link*, or a *basing of* K. Thus, a basing of K determines elements $\{\mu_i, \lambda_i\} \subset A(K)$. Note that we have, alternatively, chosen a set of meridian-longitude pairs $\{x_i, l_i\} \subseteq \pi_1(X, x_0)$ and set $\mu_i = \Phi(x_i)$, $\lambda_i = \Phi(l_i)$, where $\Phi: \pi_1(X, x_0) \to A(K)$ is defined in [6, p. 48].

Let S be the multiplicative subset of Λ consisting of all $\lambda \in \Lambda$ satisfying $\varepsilon(\lambda) = 1$. We consider the localizations Λ_S and $A_S = A \otimes_\Lambda \Lambda_S$. It is proved in [4] that A_S is an invariant of the I-equivalence class of K. Abusing notation, we will denote by $\{\mu_i, \lambda_i\}$ the images of these elements in A_S under the canonical homomorphism $A \to A_S$, and $e: A_S \to \Lambda_S$ the obvious extension of e defined above.

We recall, from [4]:

Theorem. *Let K be a based link. Then*

(a) *A_S is generated by the elements $\mu_1, \dots, \mu_n$.*

(b) *If we write, for some $a_{ij} \in \Lambda_S$,*

$$\lambda_i = \sum_{j=1}^{n} a_{ij}\mu_j \quad \text{in } A_S, \quad i = 1, \dots, n, \tag{1}$$

then A_S has a presentation with generators $\{\mu_1, \dots, \mu_n\}$ and relations

$$(T_i - 1)\mu_i = (t_i - 1) \sum_{j=1}^{n} a_{ij}\mu_j, \quad i = 1, \dots, n. \tag{2}$$

Remarks. The relations (2) can be derived from the obvious relation $[x_i, l_i] = 1$ in $\pi_1(X, x_0)$, by applying Φ and using the multiplicative property given in [6, eq. (12)]. Any one of the relations in (2) is a consequence of the remaining relations. Note that, by applying e to (2) we derive the equation

$$T_i - 1 = \sum_{j=1}^{n} a_{ij}(t_j - 1), \quad i = 1, \ldots, n. \tag{3}$$

We record the ambiguity of $a = (a_{ij})$.

(i) If the basing of K is fixed, we may add to the representation $\sum_{j=1}^{n} a_{ij}\mu_j$, of λ_i, any linear combination of the elements $\{(T_i - 1)\mu_i - (t_i - 1)\sum a_{ij}\mu_j\}$. Thus a may be changed to a' given by

$$a' = a + c(\gamma - \Delta a) \tag{4}$$

where c is *any* $(n \times n)$-matrix over Λ_S, $\gamma = \mathrm{diag}(T_1 - 1, \ldots, T_n - 1)$, and $\Delta = \mathrm{diag}(t_1 - 1, \ldots, t_n - 1)$.

(ii) If a different basing of K is chosen, with arcs $\{a_i'\}$, then the elements $\{\mu_i, \lambda_i\}$ are replaced by $\{\mu_i', \lambda_i'\}$ given by

$$\mu_i' = g_i\mu_i + (1 - t_i)\alpha_i, \qquad \lambda_i' = g_i\lambda_i + (1 - T_i)\alpha_i,$$

where α_i is represented by the lift of the composite path $a_i' \cdot a_i^{-1}$ starting at $\tilde{x}_0$, and g_i is the monomial of Λ given by $g_i = e(\alpha_i) + 1$. A simple computation shows that a is replaced by:

$$a' = (ga - \gamma c)(g - \Delta c)^{-1} \tag{5}$$

where $g = \mathrm{diag}(g_1, \ldots, g_n)$, and c is an $(n \times n)$-matrix over Λ_S satisfying: $\alpha_i = \sum c_{ij}\mu_j$ in A_S. There is the relation:

$$g_i - 1 = \sum_{j=1}^{n} c_{ij}(t_j - 1), \quad i = 1, \ldots, n. \tag{5a}$$

Note that we can effect such a change for any $\{g_i, c_{ij}\}$ satisfying (5a), as long as $c_{ij} \in \Lambda$, but the possible general choices for $\{c_{ij}\}$ depend on knowing which linear combinations $\sum_{i=1}^{n} c_i\mu_i$ belong to A.

Given a based link K, we will refer to any matrix a given by (1) as a *longitude matrix* for K.

§ 2

We will now show that the longitude matrix a can be chosen to satisfy a Hermitian condition. For simplicity *we assume, in Sections* 2–4, *that all* $l_{ij} = 0$.

Define an $(n \times n)$-matrix $\sigma = (\sigma_{ij})$, over Λ, as follows:

$$\sigma_{ij} = \begin{cases} (t_i - 1)(1 - t_j^{-1}), & j > i, \\ (t_i - 1), & i = j, \\ 0, & i > j. \end{cases} \tag{6}$$

We note the following properties of σ:

$\sigma = \sigma_0\Delta = \Delta\sigma_1$, where σ_0, σ_1 are suitably chosen matrices over Λ which are invertible over Λ_S. (7)

If $g \in \Lambda_S$, then $\bar{g}$ will denote the image of g under the anti-automorphism of Λ_S defined by $t_i \mapsto t_i^{-1}$. If $b = (b_{ij})$ is any matrix over Λ_S, then $\bar{b} = (\bar{b}_{ij})$ and $b^{\mathrm{T}} = (b_{ji})$, the transpose of b.

$$\sigma + \bar{\sigma}^{\mathrm{T}} = -\rho\bar{\rho}^{\mathrm{T}} \text{ where } \rho \text{ is the column-vector } \begin{pmatrix} t_1 - 1 \\ \vdots \\ t_n - 1 \end{pmatrix}. \tag{8}$$

Note that $a\rho = 0$ (this is just (3)) and so:

$$a\sigma = -a\bar{\sigma}^{\mathrm{T}}. \tag{9}$$

Theorem A. *Given any based link K one can choose a longitude matrix a which satisfies the equation*

$$\bar{a}^{\mathrm{T}} - a + a\sigma\bar{a}^{\mathrm{T}} = 0. \tag{10}$$

If we further localize Λ we may obtain a better Hermitian property. Let $\bar{S} = \{\lambda(t_1, \ldots, t_n) \colon \pm\lambda(1, \ldots, 1) \text{ is a power of } 2\}$.

Corollary. *Given any based link, one can choose a longitude matrix over $\Lambda_{\bar{S}}$ which satisfies $\bar{a}^{\mathrm{T}} = a$.*

Proof of Corollary. Rewrite (10) as

$$a(1 - \tfrac{1}{2}\sigma\bar{a}^{\mathrm{T}}) = (1 + \tfrac{1}{2}a\sigma)\bar{a}^{\mathrm{T}}$$

or, using (9),

$$(1 + \tfrac{1}{2}a\sigma)^{-1}a = \bar{a}^{\mathrm{T}}(1 + \tfrac{1}{2}\bar{\sigma}^{\mathrm{T}}\bar{a}^{\mathrm{T}})^{-1}. \tag{10a}$$

But $(1 + \frac{1}{2}a\sigma)^{-1}$ can be written as $1 - c\Delta$ where $c = \frac{1}{2}a\sigma_0(1 + \frac{1}{2}\Delta a\sigma_0)^{-1}$, using (7). Now $a' = (1 - c\Delta)a$ is another longitude matrix for K, obtained by a transformation of the form (4). The required Hermitian property for a' is given by (10a). □

An interesting consequence of (10) is the formula

$$\sum_i a_{ij}(t_i^{-1} - 1) = 0, \quad j = 1, \ldots, n. \tag{10b}$$

To see this, consider the equation

$$0 = \bar{\rho}^{\mathrm{T}}(\bar{a}^{\mathrm{T}} - a + a\sigma\bar{a}^{\mathrm{T}}) = -\bar{\rho}^{\mathrm{T}}a(1 - \sigma\bar{a}^{\mathrm{T}})$$

where ρ is defined in (8) and we use $a\rho = 0$. Since $1 - \sigma\bar{a}^{\mathrm{T}}$ is non-singular we have $\bar{\rho}^{\mathrm{T}}a = 0$, which coincides with (10b).

Finally, we examine the consequences of Theorem A for 2-component links.

Corollary. *For any based 2-component link with zero linking number, A_S is generated by μ_1, μ_2 subject to the single relation*

$$(x-1)(y-1)^2\phi\mu_1=(x-1)^2(y-1)\phi\mu_2$$

for some $\phi\in\Lambda_S$ satisfying $\phi=\bar{\phi}$. Furthermore, for any such ϕ there is a link realizing this Alexander module.

Remark. The Alexander Polynomial is $(x-1)(y-1)\phi$. The self-conjugacy of ϕ is a classical result of [12], while the realizability is proven in [1].

Proof. The equations (3) and (10b) imply, in a straightforward manner, that there exists $\psi\in\Lambda_S$ so that

$$a_{11}=(y-1)(y^{-1}-1)\psi,\qquad a_{12}=(x-1)(1-y^{-1})\psi,$$

$$a_{21}=(x^{-1}-1)(1-y)\psi,\qquad a_{22}=(x-1)(x^{-1}-1)\psi.$$

Since Δa is a presentation matrix for A_S, we obtain the desired relation except that ϕ is replaced by ψ. In fact ψ is not self-conjugate.

If we substitute the given values of a_{ij} in the matrix equation (6), we obtain the following equation:

$$\bar{\psi}(1+(x-1)(y-1)\psi)=\psi(1+(x^{-1}-1)(y^{-1}-1)\bar{\psi}).$$

If we define

$$\phi=\psi(1+(x-1)(y-1)\psi)^{-1},$$

then ψ can be replaced by ϕ in the relation, and ϕ is self-conjugate.

§ 3

As a preliminary to the proof of Theorem A, we make a digression.

Let Σ denote a closed oriented surface of genus n. $H_1(\Sigma)$ is free abelian of rank $2n$ and we may choose a basis $\{m_1,\ldots,m_n,l_1,\ldots,l_n\}$ where m_i, l_i is a meridian, longitude pair of the ith handle, pair-wise disjoint, except that m_i intersects l_i in the point s_i, oriented so that the intersection numbers are $m_i\cdot l_i=1=-l_1\cdot m_i$. Consider the regular covering $\tilde{\Sigma}\to\Sigma$ defined by the epimorphism $H_1(\Sigma)\to Z^m$, $m_i\to t_i$, $l_i\to 0$, where $\{t_1,\ldots,t_n\}$ is a basis of Z^m. We compute $H_1(\tilde{\Sigma})$ and the intersection pairing $H_1(\tilde{\Sigma})\times H_1(\tilde{\Sigma})\to\Lambda$.

Let $s\in\Sigma$ be a basepoint disjoint from all the meridian, longitude curves, and $\tilde{s}$ the fiber over s in $\tilde{\Sigma}$. $H_1(\tilde{\Sigma},\tilde{s})$ is a Λ-module with generators $u_1,\ldots,u_n,v_1,\ldots,v_n$ and the single relation $\sum_{i=1}^n(t_i^{-1}-1)v_i=0$, where the generators can be given explicit representatives as follows. Choose oriented arcs a_i in Σ from s to s_i, which should be disjoint from each other and the meridian, longitude curves, except at the end points. Choose a basepoint $\bar{s}\in\tilde{s}$. Let $\bar{a}_i$ be the lift of a_i starting at $\bar{s}$, and let $\bar{s}_i$ denote its end point. Let $\bar{m}_i$ be the lift of m_i starting at $\bar{s}_i$, and $\bar{l}_i$ the lift of l_i which intersects $\bar{m}_i$. Then v_i is to be represented by $\bar{l}_i$ and u_i represented by $\bar{m}_i+(1-t_i)\bar{a}_i$. $H_1(\tilde{\Sigma})$ imbeds in $H_1(\tilde{\Sigma}, \tilde{s})$ as the submodule generated by $\{v_i\}$ and $\{u_{ij}=(t_i-1)u_j-(t_j-1)u_i\}$.

To explicitly compute intersection numbers we put some more restrictions on the a_i. Consider a small disk D in Σ with s in its interior whose boundary C is crossed by each a_i once. We ask that these intersection points occur in order $a_1, \ldots, a_n$ as we travel around C clockwise. Furthermore we ask that the normal direction to a_i defined by m_i at s_i agree with the normal direction to a_i defined by counterclockwise orientation of C. (See Fig. 1.)

Lemma A. *If the generators $\{u_i, v_i\}$ of $H_1(\tilde{\Sigma}, \tilde{s})$ are chosen as in the previous paragraphs, then the intersection pairing: $\langle\ ,\ \rangle: H_1(\tilde{\Sigma})\times H_1(\tilde{\Sigma})\to\Lambda$ is induced by the following pairing defined on the free module F on $\{u_i, v_i\}$:*

$$\langle u_i, u_j\rangle=\sigma_{ij} \quad \text{(defined in Section 2)},$$

$$\langle u_i, v_j\rangle=\delta_{ij}=-\langle v_i, u_j\rangle, \qquad \langle v_i, v_j\rangle=0.$$

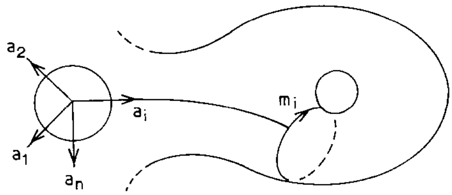

Fig. 1.

Proof. The only part that is not immediate is the value of the intersection numbers $\langle u_{ij}, u_{kl}\rangle$. To simplify this, let $\Sigma_0=\overline{\Sigma-D}$ and consider $H_1(\tilde{\Sigma}_0, \partial\tilde{\Sigma}_0)=H_1(\tilde{\Sigma}, \tilde{s})$ and $H_1(\tilde{\Sigma}_0)$ the submodule of F generated by $\{u_{ij}, v_i\}$. It suffices to consider the intersection pairing $H_1(\tilde{\Sigma}_0)\times H_1(\tilde{\Sigma}_0, \partial\tilde{\Sigma}_0)\to\Lambda$ and verify that $\{u_{ij}, u_k\}$ agree with the values claimed in the lemma.

We represent $u_{ij}\in H_1(\tilde{\Sigma}_0)$ and $u_k\in H_1(\tilde{\Sigma}_0, \partial\tilde{\Sigma}_0)$ as follows. Let u_i', u_i'' be the arcs representing $u_i\in H_1(\tilde{\Sigma}_0, \partial\tilde{\Sigma}_0)$ as illustrated in Fig. 2, with end points y_i, z_i, y_i', z_i'.

Let b_i be the arc in C from y_i to z_i and c_{ij} the arc from z_i to y_j, if $i<j$, as illustrated in Fig. 3.

Let $\bar{D}$ be the lift of D containing $\bar{s}$, and $\bar{C}$ its boundary.

Let $\bar{u}_i'$, $\bar{u}_i''$ be the lifts of u_i', u_i'' which start on $\bar{C}$ and $\bar{b}_i$, $\bar{c}_{ij}$ be the lifts of b_i, c_{ij} in $\bar{C}$. We can represent u_{ij} $(i<j)$ in $H_1(\tilde{\Sigma}_0)$ by the cycle

$$(t_i-1)\bar{u}_j'-(t_j-1)\bar{u}_i'-(t_i-1)(t_j-1)\bar{c}_{ij}+(t_j-1)\bar{b}_i-t_j(t_i-1)\bar{b}_j$$

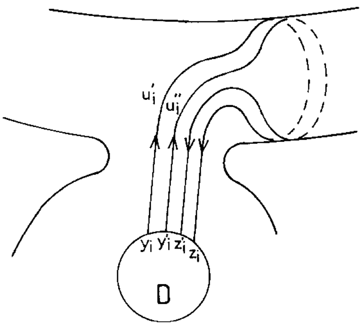

Fig. 2.

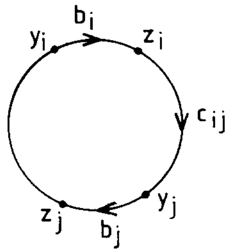

Fig. 3.

and u_i in $H_1(\tilde{\Sigma}_0, \partial\tilde{\Sigma}_0)$ by $\bar{u}_i''$. If we observe the intersection numbers

$$\langle \bar{b}_i, \bar{u}_j'' \rangle = \delta_{ij}(t_i^{-1} - 1),$$

$$\langle \bar{c}_{ij}, \bar{u}_k'' \rangle = \begin{cases} t_k^{-1} - 1 & \text{if } i < k < j, \\ 0 & \text{otherwise,} \end{cases}$$

it is straightforward to compute the intersection numbers $\langle u_{ij}, u_k \rangle$ and check that this agrees with the conclusion of Lemma A.

Note that $\langle\ ,\ \rangle$ is *not* skew-Hermitian on all of F but one can check that the restriction to $H_1(\tilde{\Sigma})$ is skew-Hermitian. This corresponds to the fact that $\langle\ ,\ \rangle$ has a *geometric* definition only on $H_1(\tilde{\Sigma})$.

§ 4

We now proceed to the proof of Theorem A.

We assume the link K is based and carry over the notation and constructions of Section 1.

Let W denote a 3-disk imbedded in X so that (see Fig. 4);

(i) $W \cap \partial N_i = \partial W \cap \partial N_i = D_i$ is a 2-disk;

(ii) ∂W contains x_0 and $a_1, \ldots, a_n$;

(iii) $m_i, l_i \subset \overline{\partial N_i - D_i}$.

Let Y denote $X - W$, and $\tilde{Y} \to Y$ the regular Z^m-covering induced by $\tilde{X} \to X$.

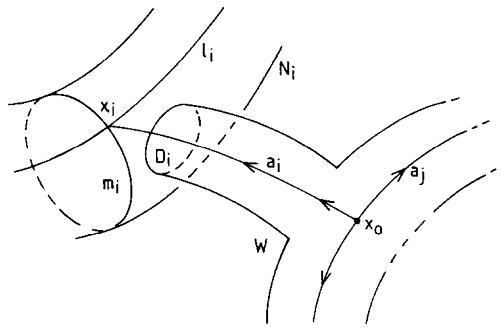

Fig. 4.

We first point out that $H_1(\tilde{Y}, \tilde{x}_0)_s$ is a free Λ_S-module with basis $\hat{\mu}_1, \ldots, \hat{\mu}_n$, where $\hat{\mu}_i$ is represented by $m_i + (1-t_i)\tilde{a}_i$. To see this, recall that, if V denotes the union of all the arcs $\{a_i\}$ and meridian circles $\{m_i\}$, then $H_1(\tilde{V}, \tilde{x}_0)$ is a free Λ-module with basis given by the loops $\{\hat{\mu}_i\}$ (see [4]). But the inclusion, $(V, x_0) \to (Y, x_0)$ induces a homology isomorphism, and therefore, by Nakayama's lemma, an isomorphism

$$H_*(\tilde{V}, \tilde{x}_0)_S \xrightarrow{\approx} H_*(\tilde{Y}, \tilde{x}_0)_S.$$

Now the closed curves $\bar{l}_i$ in $\partial\tilde{Y}$ represent classes $\hat{\lambda}_i \in H_1(\tilde{Y})_S \hookrightarrow H_1(\tilde{Y}, \tilde{x}_0)_S$ and, therefore, we may write $\hat{\lambda}_i = \sum_{j=1}^n a_{ij}\hat{\mu}_j$ $(i = 1, \ldots, n)$ in $H_1(\tilde{Y}, \tilde{x}_0)_S$. Since $Y \subset X$, we obviously have $\lambda_i = \sum_{j=1}^n a_{ij}\mu_j$ $(i = 1, \ldots, n)$ and, so, $a = (a_{ij})$ is a longitude matrix for the based link.

Consider ∂Y, which is a closed orientable surface of genus n. We identify ∂Y with Σ in Section 3. We may assume W chosen so that the arcs $\{a_i\}$ satisfy the conditions of Section 3. The generators $\{u_i, v_i\}$ of $H_1(\tilde{\Sigma}, \tilde{s})$ in Section 3 are then mapped to $\{\hat{\mu}_i, \hat{\lambda}_i\}$ in $H_1(\tilde{Y}, \tilde{x}_0)$ under the inclusion $(\tilde{\Sigma}, \tilde{s}) = (\partial\tilde{Y}, \tilde{x}_0) \subset (\tilde{Y}, \tilde{x}_0)$. Therefore the homology classes $\xi_i = v_i - \sum_{j=1}^n a_{ij}u_j \in H_1(\partial\tilde{Y}, \tilde{x}_0)_S$ are null-homologous in $H_1(\tilde{Y}, \tilde{x}_0)_S$. In fact $\xi_i \in H_1(\partial\tilde{Y})_S$ is null-homologous in $H_1(\tilde{Y})_S$. As a consequence, every intersection number $\langle \xi_i, \xi_j \rangle = 0$ (all i, j). We compute these intersection numbers using Lemma A.

$$\begin{aligned}
\langle \xi_i, \xi_j \rangle &= \left\langle v_i - \sum_{r=1}^n a_{ir}u_r,\ v_j - \sum_{s=1}^n a_{js}u_s \right\rangle \\
&= \langle v_i, v_j \rangle - \sum_r a_{ir}\langle u_r, v_j \rangle - \sum_s \bar{a}_{js}\langle v_i, u_s \rangle + \sum_{r,s} a_{ir}\bar{a}_{js}\langle u_r, u_s \rangle \\
&= 0 - a_{ij} + \bar{a}_{ji} + \sum_{r,s} a_{ir}\bar{a}_{js}\sigma_{rs}.
\end{aligned}$$

This is the formula claimed in Theorem A.

§ 5

We now examine the effect of a surgery modification on a link. We carry over our usual notation from Section 1: K will denote a based link of n-components; X the complement of an open tubular neighborhood of K; $\{a_i\}$ a choice of arcs defining the given basing of K, from the base-point $x_0 \in X$ to the boundary components; $Y \subset X$ the complement of a smooth regular neighborhood of $K \cup \{a_i\}$; $\{\lambda_i, \mu_i\}$ elements of $A(K) = H_1(\tilde{X}, \tilde{x}_0)$ defined, using $\{a_i\}$; and the longitude matrix $a = (a_{ij})$ consisting of the coefficients in Λ_S of the equation

$$\lambda_i = \sum_{i=1}^{n} a_{ij}\mu_j, \quad i = 1, \ldots, n.$$

Suppose $B = \{B_1, \ldots, B_k\}$ is a collection of disjoint smooth circles imbedded in S^3 satisfying:

(i) $B_i \cap K_j = B_i \cap a_j = \emptyset$, for every i, j,

(ii) $l(B_i, K_j) = 0$ for every i, j,

(iii) $\{B_i\}$, considered as a link in S^3, is trivial.

The *surgical modification* K_B of the based link K along $\{B_i\}$ is defined as follows. Using a normal framing which winds around each B_i once (there are two such framings), do a surgery on S^3 along $\{B_i\}$. This converts S^3 into a new manifold Σ^3, which, by (iii), is again diffeomorphic to S^3. But $\{K_i, a_i\}$ lie in Σ^3 and can be considered to define the based link K_B in Σ^3.

Another description of K_B can be given as follows. Choose disjoint imbedded disks $\{D_i\}$ in S^3 bounded by $\{B_i\}$—we may assume that K and the arcs $\{a_i\}$ intersect each D_i transversely, and $D_i \cap K_j \cap a_j = \emptyset$, for each i, j. Now cut the strands of $\{K, a_i\}$ along each D_j and give each D_j a 360° rotation (in either direction) after which connect the strands again (see Fig. 5). The resulting link is K_B.

Our main interest here is to describe a longitude matrix for K_B from one on K.

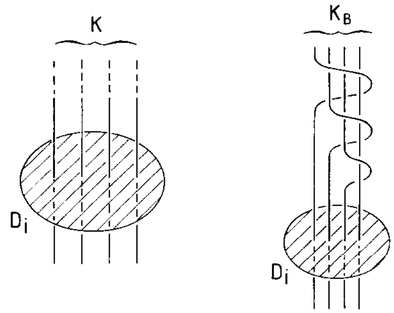

Fig. 5.

For this purpose, choose an oriented lift $\bar{B}_i$ of B_i into $\tilde{Y}$ (note $B_i \subset Y$ by (i)) and write $\beta_i = \sum b_{ij}\hat{\mu}_j$, $b_{ij} \in \Lambda_S$, where $\beta_i \in H_1(\tilde{Y}, \tilde{x}_0)_S$ is represented by $\bar{B}_i$—note that $\bar{B}_i$ is a closed curve, by (ii), which also means $e(\beta_i) = 0$. This yields

$$\sum_{j=1}^{n} b_{ij}(t_j - 1) = 0, \quad i = 1, \ldots, k. \tag{11}$$

Theorem B. *If the based link K has a longitude matrix a, then the based link K_B has a longitude matrix a' given by*

$$a' = a - (1 + a\sigma)\bar{b}^{\mathrm{T}} c^{-1} b \tag{12}$$

where σ is the matrix defined in Section 2 *and $c = (c_{ij})$ is a $(k \times k)$-matrix, over Λ_S, with the properties*

$$\varepsilon(c_{ij}) = \pm\delta_{ij}, \tag{13}$$

where $\varepsilon : \Lambda_S \to Z$ is the augmentation defined by $t_i \to 1$

$$c - \bar{c}^{\mathrm{T}} = b\sigma\bar{b}^{\mathrm{T}}. \tag{14}$$

Moreover, given any matrix b over Λ_S satisfying (11), *and c, over Λ_S, satisfying* (13) *and* (14), *there is a surgical modification of K which yields* (12) *as a longitude matrix.*

Remarks. (i) Note that (13) implies that c is invertible over Λ_S.

(ii) It is easy to check that a' satisfies (10), if a does.

Proof. Let $X_0(Y_0)$ denote the manifold obtained by removing disjoint open tubular neighborhoods of $\{B_i\}$ from $X(Y)$. Then $H_1(\tilde{Y}_0, \tilde{x}_0)_S$ is the free Λ_S-module with basis $\bar{\mu}_1, \ldots, \bar{\mu}_n, \varepsilon_1, \ldots, \varepsilon_k$, where $\bar{\mu}_i \in H_1(\tilde{Y}_0, \tilde{x}_0)$ is represented by the same chain which represents $\hat{\mu}_i$ in $H_1(\tilde{Y}, \tilde{x}_0)$ and $\varepsilon_i \in H_1(\tilde{Y}_0, \tilde{x}_0)$ is represented by a meridian curve around $\bar{B}_i$. The classes $\bar{\lambda}_i \in H_1(\tilde{Y}_0, \tilde{x}_0)$, represented by l_i, can be expressed, in $H_1(\tilde{Y}_0, \tilde{x}_0)_S$, as linear combinations

$$\bar{\lambda}_i = \sum_{j=1}^{n} a_{ij}\bar{\mu}_j + \sum_{j=1}^{k} r_{ij}\varepsilon_j, \quad i = 1, \ldots, n. \tag{15}$$

Let $\alpha_i \in H_1(\tilde{Y}_0, \tilde{x}_0)$ be represented by the translate $\bar{B}_i'$ of $\bar{B}_i$ along one of the vector fields in the framing. We may represent α_i, in $H_1(\tilde{Y}_0, \tilde{x}_0)_S$ by a linear combination:

$$\alpha_i = \sum_{j=1}^{n} b_{ij}\bar{\mu}_j + \sum_{j=1}^{k} c_{ij}\varepsilon_j, \quad i = 1, \ldots, k. \tag{16}$$

The completion of the surgery fills in the holes created by removing the tubular neighborhoods of $\{B_i\}$, so as to add the relations $\alpha_i = 0$ to $H_1(\tilde{Y}_0, \tilde{x}_0)$. Assuming that $c = (c_{ij})$, from (16), is invertible, we can solve the equations $\alpha_i = 0$, using (16), for $\{\varepsilon_i\}$ and substitute into (15). If we use $\{\hat{\lambda}_i', \hat{\mu}_i'\}$ to indicate the classes which $\{\bar{\lambda}_i, \bar{\mu}_i\}$ become, after completion of the surgery we now have $\hat{\lambda}_i' = \sum_{j=1}^n a_{ij}' \hat{\mu}_j'$, $i = 1, \dots, n$, where

$$a' = a - rc^{-1}b. \tag{17}$$

In $H_1(\partial\tilde{Y}_0)$ consider the classes

$$\begin{aligned} \eta_i &= \sum_{j=1}^n b_{ij}u_j + \sum_{j=1}^k c_{ij}e_j - f_i, \quad i = 1, \dots, k, \\ \xi_i &= \sum_{j=1}^n a_{ij}u_j + \sum_{j=1}^k r_{ij}e_j - v_i, \quad i = 1, \dots, n, \end{aligned} \tag{18}$$

where $\{u_i, v_i\}$ and $\{e_i, f_i\}$ are meridian, longitude pairs in $H_1(\partial\tilde{Y}_0, \tilde{x}_1)$ which map to $\{\bar{\mu}_i, \bar{\lambda}_i\}$ and $\{\varepsilon_i, \alpha_i\}$ in $H_1(\tilde{Y}_0, \tilde{x}_0)$ under inclusion (followed by a path translation from base-point $\tilde{x}_1 \in \partial\tilde{Y}_0$ to $\tilde{x}_0$). It follows that the intersection numbers

$$\{\xi_i, \xi_j\rangle = \langle\eta_i, \eta_j\rangle = \langle\xi_i, \eta_j\rangle = 0.$$

We expand out these equations using (18) and

$$\langle u_i, v_j\rangle = \delta_{ij} = \langle e_i, f_j\rangle, \qquad \langle v_i, v_j\rangle = \langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0,$$

$$\langle e_i, u_j\rangle = \langle f_i, u_j\rangle = \langle e_i, v_j\rangle = \langle f_i, v_j\rangle = 0,$$

which are all obvious, and $\langle u_i, u_j\rangle = \sigma_{ij}$, from Lemma A.

The result of expanding out $\langle\xi_i, \eta_j\rangle = 0$, for all i, j, is:

$$a\sigma\bar{b}^{\mathrm{T}} + \bar{b}^{\mathrm{T}} - r = 0$$

and so $r = (1 + a\sigma)\bar{b}^{\mathrm{T}}$. Substituting this into (17) gives (12).

The result of expanding out $\langle\eta_i, \eta_j\rangle = 0$, for all i, j is

$$b\sigma\bar{b}^{\mathrm{T}} - c + \bar{c}^{\mathrm{T}} = 0, \quad \text{i.e. (14).}$$

Finally, to prove (13), we project (16) down to give an equation in $H_1(Y_0)$

$$\gamma_i = \sum_{j=1}^n \varepsilon(b_{ij})\nu_j + \sum_{j=1}^k \varepsilon(c_{ij})\tau_j, \quad i = 1, \dots, k$$

where γ_i is represented by B_i', ν_j is represented by m_j and τ_j represented by a meridian about B_i. Since $H_1(Y_0)$ is the free abelian group generated by $\{\nu_i, \tau_i\}$, we may interpret $\varepsilon(c_{ij})$ as the linking number of B_i' and B_j. But by our choices, this is $\pm\delta_{ij}$.

We now have the task of finding a surgical modification which realizes any a' given by (12), for suitable b, c.

We first consider the special case where b and c are matrices with entries *in* Λ. Since $\hat{\mu}_i \in H_1(\tilde{Y}, \tilde{x}_0)$, we can choose $\bar{B}_i$, a closed curve in $\tilde{Y}$, to represent $\sum_{j=1}^n b_{ij}\hat{\mu}_j$ in $H_1(\tilde{Y}, \tilde{x}_0)$ and we can assume the $\{\bar{B}_i\}$ project to disjoint simple closed curves $\{B_i\}$ in Y. We can even arrange that $\{B_i\}$ satisfy (iii)—this follows by the same construction used in [5, § 10]. Choose the normal frame to each B_i so that $l(B_i, B_i') = \varepsilon(c_{ii})$. We can now do surgery along $\{B_i\}$ to obtain K_B which will have a longitude matrix a' given by (12), with the given b and *some* c. In fact c will have entries in Λ, since we may choose a representation (16) of α_i as an element of $H_1(\tilde{Y}_0, \tilde{x}_0)$—note that kernel $\{H_1(\tilde{Y}_0, \tilde{x}_0) \to H_1(\tilde{Y}, \tilde{x}_0)\}$ is generated by $\{\varepsilon_i\}$. To realize the desired c, we will have to change the c which arises from the construction by $c \mapsto c+d$, where d is an arbitrary matrix over Λ satisfying $\varepsilon(d)=0$ and $d=\bar{d}^{\mathrm{T}}$—since both c and $c+d$ satisfy (13) and (14) (and $\varepsilon(c_{ij})$ is correct, already, for every (i). The method for doing this is exactly the same as given in [6, appendix].

We now reduce the general case, where b, c have entries in Λ_S to this special case. Choose an element $s \in S$ with the property that $sb_{ij}, sc_{ij} \in \Lambda$ for every i, j. Now define $b'_{ij} = sb_{ij}$ and $c'_{ij} = s\bar{s}c_{ij}$. It is easy to check that b' satisfies (11), c' satisfies (13) and b', c' satisfy (14). By the special case, we can construct a surgical modification of K which has longitude matrix a' given by (12), where b, c are replaced by b', c'.

But $\bar{b}'^{\mathrm{T}}(c')^{-1}b' = \bar{b}^{\mathrm{T}}c^{-1}b$, and so we have, in fact, realized a' given by (12) for b, c. □

Remark. Theorem B suggests the following interpretation.

Let $a'' = -b^{\mathrm{T}}c^{-1}b$; a straightforward computation shows that a'' satisfies equation (10). Thus a'' resembles a longitude matrix for a based link with all linking numbers zero. In fact, the special case $a=0$ in Theorem B shows that a'' is the longitude matrix for a surgical modification of the trivial link.

Suppose a, a' are longitude matrices for based links K, K', which satisfy (10). Then one can compute that $a'' = a + a' + a\sigma a'$ also satisfies (10). It seems likely that a'' is a longitude matrix for the connected sum K'' of K and K', which will be well-defined in the context of based links.

§ 6

Finally, we discuss the problem of characterizing the Alexander polynomial of a link—but localized in Λ_S.

Recall the definition of the Alexander polynomial Δ of a link as a greatest common divisor of the Fitting ideal $\mathcal{O}_1$ of the Alexander module. Note that Δ is only defined up to multiplication by a unit of Λ. In fact, we have (for $n>1$) the equality: $\mathcal{O}_1 = \Delta I$, where I is the augmentation ideal (see [3]). We remind the reader that the *Fitting ideal* $\mathcal{O}_i$ $(i=0,1,2,\ldots)$ of a finitely-generated module over a commutative ring is the ideal generated by all $(n-i)\times(n-i)$ minors of any presentation matrix (a_{ij}) corresponding to a presentation: $\{\alpha_1, \ldots, \alpha_n : \sum_{j=1}^n a_{ij}\alpha_j = 0,\ i=1,\ldots,m\}$ with n generators.

In [12] a set of conditions, satisfied by a canonical choice of the Alexander polynomial, are established and these can be made the basis of a conjecture on which polynomials arise as Alexander polynomials (see e.g. [5]). For example, when

$n=2$, the Torres conditions on $\Delta(x,y)$ are the following (l = linking number of the two components).

(a) $\Delta(x^{-1},y^{-1})=(xy)^{1-l}\Delta(x,y)$,

(b) $\Delta(x,1)=\dfrac{x^l-1}{x-1}\phi_1(x), \qquad \Delta(1,y)=\dfrac{y^l-1}{y-1}\phi_2(y)$,

where $\phi_1(x)$, $\phi_2(y)$ are the Alexander polynomials of the knots defined by the individual components. Since these knot polynomials can be characterized by the conditions $\phi_1(x)=\phi_1(x^{-1})$, $\phi_1(1)=\pm1$, we may, without loss of information, rewrite (b)

(b)(i) $\Delta(1,1)=\pm l$,

(ii) $\Delta(\omega,1)=0=\Delta(1,\omega)$, where ω is any lth root of unity except 1.

In [2], a polynomial is exhibited (for $l=6$) which satisfies (a), (b) but is not the Alexander polynomial of any link. In [10], such polynomials are exhibited for any $l\neq 0,1,2$. It is known that (a), (b) do characterize the Alexander polynomial for $l=0,1$; $l=2$ is still unresolved.

We can consider the localized Alexander polynomial by using Λ_S instead of Λ. This is, of course, the same as the ordinary Alexander polynomial, except that we consider it as well-defined only up to multiplication (or division) by an element of S.

Theorem C. *An element of Λ is the (localized) Alexander polynomial of a link if and only if it satisfies* (a), (b).

In other words, Theorem C asserts that, given any polynomial Δ satisfying (a), (b) then there exists some $\phi\in S$ such that $\Delta\cdot\phi$ is the Alexander polynomial of a link. Note that, for $\Delta\cdot\phi$ to satisfy (a), it is necessary that $\phi(x,y)=\phi(x^{-1},y^{-1})$. Compare this to the result [4] that if Δ is already the Alexander polynomial of a link, and $\phi\in S$ satisfies $\phi(x,y)=\phi(x^{-1},y^{-1})$, then $\Delta\cdot\phi$ is again the Alexander polynomial of a link.

Proof of Theorem C. This will be a direct application of the main result of [1], which we recall, in the following form:

Theorem [1]. *A polynomial $\Delta(x,y)$ satisfying* (a), (b) is the Alexander polynomial of a link with linking number l if and only if we can write

$$\Delta(x,y)=\Phi_l(x,y)A(x,y)-(x-1)(y-1)\Phi_{l-1}(x,y)B(x,y) \tag{19}$$

where $\Phi_r(u)=(u^r-1)/(u-1)$, and A, B are the determinants of matrices $\boldsymbol{A}$, $\boldsymbol{B}$, over Λ, which satisfy

$$\boldsymbol{B}=\begin{bmatrix}0 & \bar{\beta}^{\mathrm{T}}\\ \beta & \boldsymbol{A}\end{bmatrix}$$

where β is a row vector, $\boldsymbol{A}=\bar{\boldsymbol{A}}^{\mathrm{T}}$ and $\varepsilon(\boldsymbol{A})=\operatorname{diag}(\pm1,\ldots,\pm1)$.

Recall also (see [1]), that a polynomial Δ satisfies (a), (b) if and only if it can be written in the form (19), where A, B satisfy

$$A(x,y)=A(x^{-1},y^{-1}), \qquad A(1,1)=\pm 1, \qquad B(x,y)=B(x^{-1},y^{-1}). \tag{20}$$

Therefore, to prove Theorem C, it suffices to show that, given A, B satisfying (20), there exists $\phi \in S$ and matrices $\boldsymbol{B}$, $\boldsymbol{A}$, as above, so that $A\cdot\phi=\det\boldsymbol{A}$, $B\cdot\phi=\det\boldsymbol{B}$.

We need the following lemma.

Lemma B. *Given any $f\in\Lambda$ satisfying $f(x,y)=f(x^{-1},y^{-1})$ there exist a finite sequence $f_1,\dots,f_k\in\Lambda$ and $\varepsilon_1,\dots,\varepsilon_k$, where each $\varepsilon_i=\pm1$, such that*

$$f(x,y)=\sum_{i=1}^{k}\varepsilon_i f_i(x,y)f_i(x^{-1},y^{-1}).$$

Proof. Since every such f is a sum of terms of the form $\pm g$, where $g(x,y)=2+x^ky^l+x^{-k}y^{-l}$, or $g=1$, it suffices to notice that, in the former case, $g(x,y)=h(x,y)h(x^{-1},y^{-1})$, where $h(x,y)=1+x^ky^l$.

Now, use Lemma B to write $B(x,y)$

$$B(x,y)=-\sum_{i=1}^{k}\varepsilon_i b_i(x,y)b_i(x^{-1},y^{-1}) \tag{21}$$

and consider the matrix

$$\boldsymbol{B}=\begin{bmatrix} 0 & \bar b_1 & \cdots & \bar b_k \\ b_1 & & & \\ \vdots & & \boldsymbol{A} & \\ b_k & & & \end{bmatrix},$$

$$\boldsymbol{A}=\operatorname{diag}\{\varepsilon_i A\}.$$

An easy computation shows that

$$\det\boldsymbol{A}=\varepsilon_1\cdots\varepsilon_k A^k,$$

$$\det\boldsymbol{B}=-\sum_{i=1}^{k}\varepsilon_1\cdots\hat\varepsilon_i\cdots\varepsilon_k A^{k-1}b_i\bar b_i,$$

or $\det\boldsymbol{B}=\varepsilon_1\cdots\varepsilon_k A^{k-1}B$ by (21).

If we take $\phi=\varepsilon_1\cdots\varepsilon_k A^{k-1}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are as required.

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